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Numerical Studies on Quasilinear and Linear Elliptic Equations

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We explore here the acceleration of convergence of iterative methods for the solution of a class of quasilinear and linear algebraic equations. The specific systems are the finite difference form of the Navier–Stokes equations and the energy equation for recirculating flows. The acceleration procedures considered are: the successive over relaxation scheme; several implicit methods; and a second-order procedure. A new implicit method—the alternating direction line iterative method—is proposed in this paper. The method combines the advantages of the line successive over relaxation and alternating direction implicit methods. The various methods are tested for their computational economy and accuracy on a typical recirculating flow situation.

The numerical experiments show that the alternating direction line iterative method is the most economical method of solving the Navier–Stokes equations for all Reynolds numbers in the laminar regime. The usual ADI method is shown to be not so attractive for large Reynolds numbers because of the loss of diagonal dominance. This loss can however be restored by a suitable choice of the relaxation parameter, but at the cost of accuracy. The accuracy of the new procedure is comparable to that of the well-tested successive overrelaxation method and to the available results in the literature.

The second-order procedure turns out to be the most efficient method for the solution of the linear energy equation.

INTRODUCTION

Numerical solutions to the Naiver–Stokes and energy equations are very popular for predicting the flow and heat transfer in separated regions. The development of these solutions involves the construction of a suitable finite difference analogue that exhibits satisfactory convergence and stability properties. The "upwind" difference scheme has now been accepted as an efficient procedure for constructing solutions to recirculating flow problems [1–4]. The second step in the process, the solution of the algebraic equations generated by the differencing procedure, is usually accomplished by some form of relaxation method. One of the severe handicaps in the application of these methods for the study of complex engineering systems with a large number of parameters, is the enormous demands of computational effort. In this paper, we concentrate on this aspect of the problem and seek methods that will provide fast and accurate solutions to the algebraic equations.

The finite-differencing procedure of elliptic partial differential equations (the equations we consider in this paper belong to this general class) yields a large, sparse matrix system. These systems are usually solved by iterative methods. Many acceleration procedures are in vogue to reduce the computational effort involved in these procedures. For linear equations, a considerable body of theoretical results has been developed to predict the behavior of the different procedures [5]. No such results are available for the quasilinear coupled system of equations that are of interest in this work. Under these conditions, it appears expedient to take recourse to an experimental study of the behavior of various methods. The primary objective of the present work is to apply various acceleration procedures to a typical recirculating flow situation and arrive at ad hoc conclusions about the relative effectiveness of the different procedures, in handling this class of problems. The energy equation, though single and linear, faces similar problems of slow convergence in separated regions. Hence, the acceleration procedures for this equation have also been studied.

The cavity problem has been chosen as the model problem for detailed study in this work. Plenty of experimental as well as computational results are available for this geometry and as such will provide adequate comparisons for the work reported in this paper.

The Cavity Problem

The problem of steady, plane, laminar flow of an incompressible constant property Newtonian fluid in a square cavity, similar to the one considered by Greenspan [2], is considered in this paper. The flow inside the cavity is induced by

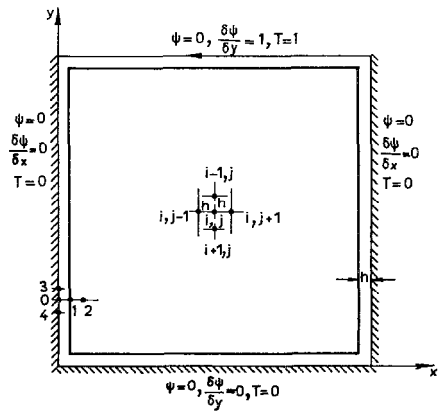

FIG. 1. Geometry and coordinates.

the steady motion of one of the walls in its own plane, while the other three are stationary. The moving wall is kept at a temperature higher than that of the other three walls which are at a uniform temperature. Figure 1 shows the geometry and coordinate system employed in the study.

The equations governing the flow inside the square cavity are:

$$\nabla^2\psi = -\omega, \tag{1}$$

$$\nabla^2\omega = Re[(\partial\psi/\partial y)(\partial\omega/\partial x) - (\partial\psi/\partial x)(\partial\omega/\partial y)]. \tag{2}$$

The boundary conditions to be satisfied are:

$$\begin{aligned} &\psi(0, y) = 0, \quad \left.\frac{\partial\psi}{\partial x}\right|_{x=0} = 0, \\ &\psi(1, y) = 0, \quad \left.\frac{\partial\psi}{\partial x}\right|_{x=1} = 0, \\ &\psi(x, 0) = 0, \quad \left.\frac{\partial\psi}{\partial y}\right|_{y=0} = 0, \\ &\psi(x, 1) = 0, \quad \left.\frac{\partial\psi}{\partial y}\right|_{y=1} = -1. \end{aligned} \tag{3}$$

The energy equation for steady, viscous, constant property fluid is

$$u(\partial T/\partial x) + v(\partial T/\partial y) = \frac{1}{Re\ Pr}[(\partial^2 T/\partial x^2) + (\partial^2 T/\partial y^2)]. \tag{4}$$

The boundary conditions are:

$$\begin{aligned} &T(x, 1) = 1, \\ &T(0, y) = T(x, 0) = T(1, y) = 0. \end{aligned} \tag{5}$$

In the above equations, ψ, ω, and T are the dimensionless stream, vorticity and temperature functions respectively; Re is the Reynolds number; Pr is the Prandtl number of the fluid; u and v are the velocity components in x and y directions.

The Finite Difference Equations

The governing differential equations are replaced by difference equations using the upwind differencing procedure [2]. These equations for any point (i, j) in the field corresponding to Eqs. (1), (2) and (4) are:

$$\psi_{i-1,j} + \psi_{i+1,j} + \psi_{i,j-1} + \psi_{i,j+1} - 4\psi_{i,j} = -h^2\omega_{i,j}, \tag{6}$$

$$\omega_{i-1,j} + \omega_{i+1,j} + \omega_{i,j-1} + \omega_{i,j+1} - 4\omega_{i,j}$$
$$= \frac{Reh}{2}[\omega_{i-1,j}(v - |v|) - \omega_{i+1,j}(v + |v|)$$
$$- \omega_{i,j-1}(u + |u|) + \omega_{i,j+1}(u - |u|) + 2\omega_{i,j}(|u| + |v|)], \tag{7}$$

$$T_{i-1,j} + T_{i+1,j} + T_{i,j-1} + T_{i,j+1} - 4T_{i,j}$$
$$= \frac{Re\,Prh}{2}[T_{i-1,j}(v - |v|) - T_{i+1,j}(v + |v|)$$
$$- T_{i,j-1}(u + |u|) + T_{i,j+1}(u - |u|) + 2T_{i,j}(|u| + |v|)], \tag{8}$$

h in the above equations is the size of the square mesh used to replace the flow field.

The next step is to replace the boundary conditions by their corresponding difference forms. The stream function presents no problem and the first set of conditions shown in Eq. (3) can be used directly. The boundary condition for vorticity or its derivative along the walls is not specified a priori, but is to be derived from the no-slip condition at the wall [2], that is, from the second set of conditions in Eq. (3). Thus, for example, on the wall $x = 0$,

$$-\nabla^2\psi|_{x=0} = \omega(0, y)$$
$$= \frac{4}{h^2}\psi_0 - \frac{2}{h^2}\psi_1 - \frac{1}{h^2}\psi_3 - \frac{1}{h^2}\psi_4 + \frac{2}{h}\frac{\partial\psi}{\partial x}\Big|_{x=0} \tag{9}$$

and

$$\frac{\partial\psi}{\partial x}\Big|_{x=0} = \frac{1}{2h}(-3\psi_0 + 4\psi_1 - \psi_2). \tag{10}$$

The subscripts on ψ refer to the points shown in Fig. 1. Similar equations for the other walls may be written. The temperature boundary condition can be used directly from Eq. (5).

The Numerical Procedure

The numerical method employed to solve the set of equations represented by (6) and (7) is an iterative one. Since the boundary condition on the vorticity function is specified in terms of unknown stream function values which are away from the boundary, special iterative procedures are required. Greenspan [2] uses an iterative

procedure consisting of an inner and an outer iteration. This procedure is illustrated with the aid of the thick line drawn at a distance of h from the boundary (see Fig. 1). The outer iteration consists of evaluating:

a. The vorticity function on all the walls from equations of the type (9);

b. the stream function on the thick line from equations of the type (10).

The inner iteration involves the evaluation of:

a. the vorticity function at all points inside the boundary from Eq. (7);

b. the stream function at all points inside the region bounded by the thick line, using Eq. (6).

For each outer iteration, Greenspan carries out the inner iteration to convergence. Instead a direct iteration procedure involving one inner iteration to every outer iteration reduces considerably the computational time according to Friedman [6]. We use the direct iteration procedure in all our investigations. The convergence criterion is defined in this work by

$$\left| \frac{f^{(n+1)} - f^{(n)}}{f^{(n+1)}} \right|_{\max} < \epsilon \tag{11}$$

where f refers to the function evaluated, n the iteration count, and ϵ an arbitrary small number.

Acceleration Procedures

Direct iterative procedures of the Gauss–Seidel type are normally slow in convergence. Many acceleration techniques have been devised from time to time to improve the rate of convergence of iterative methods. A few of them are discussed below.

Successive over Relaxation Method (SOR)

This is a point-by-point iterative scheme, similar to the Gauss-Seidel scheme, but with a relaxation factor "r" to find the new value at a point in the field. The procedure for the evaluation of a function "f" iteratively is defined by

$$\bar{f}^{(n+1)} = f^{(n)} + r(f^{(n+1)} - f^{(n)}). \tag{12}$$

$f^{(n+1)}$, in the above, is the value found explicitly from known values at the neighboring points at the $(n + 1)$th iteration and $\bar{f}^{(n+1)}$ is the updated value. The rate of

convergence of the procedure is a strong function of "r" and for linear equations, an optimum value of "r", which provides convergence rates that are faster by an order of magnitude than the Gauss–Seidel scheme ($r = 1.0$), is shown to exist. Further, simple linear equations are amenable to exact analytical treatment to derive the optimum relaxation parameters [5]. No such results are available to determine the optimum relaxation parameter for the type of equations considered in this paper. Thus, one is forced to rely upon a scanning procedure, that is, to vary r over a range, and select that value of r which provides the fewest number of iterations. The procedure is carried out for the present set of equations with $Re = 1000$, $Pr = 1.0$, $h = 0.05$ and $\epsilon = 0.0001$.

The scanning is accomplished in two stages. At first the relaxation parameter for vorticity is kept constant at a value of unity and only the value of r for stream function is varied. The result of this investigation is shown in Fig. 2 as a plot of convergence rate versus the relaxation parameter. From this figure, it is seen that the optimum relaxation parameter is 1.6. For $r_\psi > 1.6$, the number of iterations required for convergence tends to increase, and around $r_\psi = 1.9$, the method diverges.

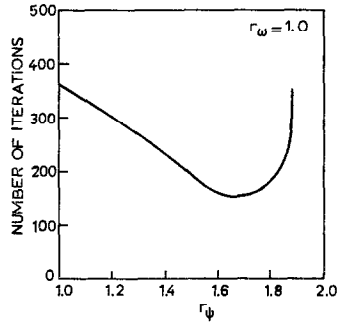

FIG. 2. Convergence rate–stream function.

Next, the relaxation parameter for stream function is kept constant at 1.6 and it is varied for the vorticity function. The result of this investigation is plotted in Fig. 3. The optimum relaxation parameter for vorticity is obtained as 1.1 from the figure. The procedure diverges for $r_\omega = 1.2$.

In order to check whether the above values of the relaxation parameters are the only ones that produce fastest convergence, the computer programmes were run for several other combinations of r_ψ and r_ω. These runs showed poorer convergence rates than those obtained with the optimum values for r_ψ and r_ω. These values of optimum relaxation parameters are insensitive to initial guesses used to start the iteration. Dynamic variation of the relaxation parameters during the iteration process is also of no help for the present system of equations. Finally, these relaxa-

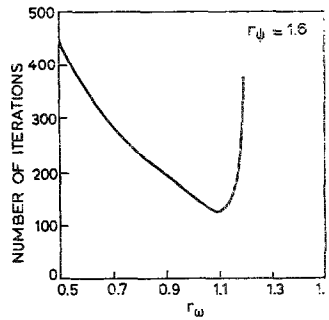

FIG. 3. Convergence rate–vorticity function.

tion parameters have been tested for a wide variety of geometries and problems [7–9] by the authors and found to work very well.

The savings in computational effort obtained by the use of these optimum relaxation parameters are considerable as the following results will reveal. While the Gauss–Seidel method ($r_\omega = r_\psi = 1.0$) takes 366 iterations for convergence, the SOR method with optimum relaxation parameters ($r_\psi = 1.6$, $r_\omega = 1.1$) takes only 115 iterations. In both cases the number of operations performed per iteration is almost the same. Thus, by the use of the SOR method with optimum relaxation parameters, the time taken for convergence is reduced to about one-third of the value with the Gauss–Seidel scheme.

A similar procedure is employed to establish the optimum relaxation parameter for the energy equation. Figure 4 shows the plot of rate of convergence against the relaxation parameter for the energy equation. The optimum relaxation parameter for this case is obtained as 1.1, and is the same as that of the vorticity function. The above result suggests that the optimum relaxation parameter is governed by the form of the equation rather than the linearity or otherwise of the equation, at least for the class of flows studied here. The reduction of time achieved by the use of the optimum relaxation parameter with the SOR method is only 17% when compared with the time taken by the Gauss–Seidel scheme.

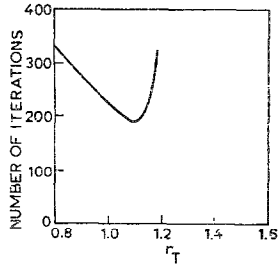

FIG. 4. Convergence rate–temperature function.

Implicit Methods

The cavity problem represents the simplest geometry in the class of steady, viscous, separated flows. Complex geometry usually results in a large system of algebraic equations. The increased size of the system usually results in an increase not only in the computational time per iteration but also in the number of iterations. It is natural in these circumstances to seek methods of further improving the rate of convergence of iterative methods. Some of these methods are implicit in nature.

In SOR method, the value of each component is found explicitly in terms of the values at the neighboring points. If it is applied to a group of points instead, there will arise a system of equations which require simultaneous solution. Consequently individual components are implicitly defined in terms of the other components of the group.

In the implicit methods, the updated value at each point is found as follows.

$$f^{(n+1)} = f^{(n)} + r(\bar{f}^{(n+1)} - f^{(n)}) \tag{13}$$

where $\bar{f}^{(n+1)}$ refers to the new value of the function evaluated implicitly at the $(n+1)$th iteration and $f^{(n+1)}$ to the updated value. The relaxation parameter is the optimum value determined earlier. These methods are treated in detail by Ames [10]. We restrict ourselves here to a brief discussion, followed by the relevant equations, for the different methods. The energy equation, being similar to that of vorticity, is not included in the following.

Line Iterative Methods

Only row and column iteration are considered in this paper. The corresponding difference equations are:

Row Iteration:

$$\bar{\psi}_{i,j}^{(n+1)} = (\psi_{i+1,j}^{(n)} + \psi_{i-1,j}^{(n+1)} + \psi_{i,j+1}^{(n+1)} + \psi_{i,j-1}^{(n+1)} + h^2\omega_{i,j}^{(n)})/4, \tag{14}$$

$$\begin{aligned}\bar{\omega}_{i,j}^{(n+1)} = \Big[&\omega_{i+1,j}^{(n)} + \omega_{i-1,j}^{(n+1)} + \omega_{i,j+1}^{(n+1)} + \omega_{i,j-1}^{(n+1)} \\ &+ \frac{Reh}{2}\{\omega_{i+1,j}^{(n)}(v + |v|) - \omega_{i-1,j}^{(n+1)}(v - |v|) \\ &- \omega_{i,j+1}^{(n+1)}(u - |u|) + \omega_{i,j-1}^{(n+1)}(u + |u|)\}\Big]/ \\ &(4 + Reh(|u| + |v|)).\end{aligned} \tag{15}$$

Iteration is carried out from the top of the cavity to the lower wall.

Column Iteration:

$$\bar{\psi}_{i,j}^{(n+1)} = (\psi_{i+1,j}^{(n+1)} + \psi_{i-1,j}^{(n+1)} + \psi_{i,j+1}^{(n)} + \psi_{i,j-1}^{(n+1)} + h^2\omega_{i,j}^{(n)})/4, \tag{16}$$

$$\begin{aligned}\bar{\omega}_{i,j}^{(n+1)} = \Big[&\omega_{i+1,j}^{(n+1)} + \omega_{i-1,j}^{(n+1)} + \omega_{i,j+1}^{(n)} + \omega_{i,j-1}^{(n+1)} \\ &+ \frac{Reh}{2}\{\omega_{i+1,j}^{(n+1)}(v + |v|) - \omega_{i-1,j}^{(n+1)}(v - |v|) \\ &- \omega_{i,j+1}^{(n)}(u - |u|) + \omega_{i,j-1}^{(n+1)}(u + |u|)\}\Big]/ \\ &(4 + Reh(|u| + |v|)).\end{aligned} \tag{17}$$

The iterations are carried out from the left to the right of the cavity.

Alternating Direction Implicit Method (ADI)

This method is somewhat similar to the line iterative method carried out alternately in x and y directions. The iterations in the x-direction are carried out by treating the x-direction derivatives as unknowns, while the y-direction iterations treat the y-direction derivatives as unknowns [10]. The difference equations corresponding to these two cases are:

Y-Direction Iteration:

$$\begin{aligned}\psi_{i,j}^{(n+1/2)} = \psi_{i,j}^{(n)} &+ r(\psi_{i+1,j}^{(n+1/2)} + \psi_{i-1,j}^{(n+1/2)} - 2\psi_{i,j}^{(n+1/2)}) \\ &+ r(\psi_{i,j-1}^{(n)} + \psi_{i,j+1}^{(n)} - 2\psi_{i,j}^{(n)} + h^2\omega_{i,j}^{(n)}),\end{aligned} \tag{18}$$

$$\begin{aligned}\omega_{i,j}^{(n+1/2)} = \omega_{i,j}^{(n)} &+ r\Big[\omega_{i+1,j}^{(n+1/2)} + \omega_{i-1,j}^{(n+1/2)} - 2\omega_{i,j}^{(n+1/2)} \\ &+ \frac{Reh}{2}\{\omega_{i+1,j}^{(n+1/2)}(v + |v|) - \omega_{i-1,j}^{(n+1/2)}(v - |v|) - 2\omega_{i,j}^{(n+1/2)}(|v|)\}\Big] \\ &+ r\Big[\omega_{i,j-1}^{(n)} + \omega_{i,j+1}^{(n)} - 2\omega_{i,j}^{(n)} \\ &+ \frac{Reh}{2}\{\omega_{i,j-1}^{(n)}(u + |u|) - \omega_{i,j+1}^{(n)}(u - |u|) - 2\omega_{i,j}^{(n)}(|u|)\}\Big].\end{aligned} \tag{19}$$

X-Direction Iteration:

$$\begin{aligned}\psi_{i,j}^{(n+1)} = \psi_{i,j}^{(n+1/2)} &+ r(\psi_{i+1,j}^{(n+1/2)} + \psi_{i-1,j}^{(n+1/2)} - 2\psi_{i,j}^{(n+1/2)} \\ &+ h^2\omega_{i,j}^{(n+1/2)}) + r(\psi_{i,j-1}^{(n+1)} + \psi_{i,j+1}^{(n+1)} - 2\psi_{i,j}^{(n+1)}),\end{aligned} \tag{20}$$

$$\omega_{i,j}^{(n+1)} = \omega_{i,j}^{(n+1/2)} + r\left[\omega_{i+1,j}^{(n+1/2)} + \omega_{i-1,j}^{(n+1/2)} - 2\omega_{i,j}^{(n+1/2)} + \frac{Reh}{2}\{\omega_{i+1,j}^{(n+1/2)}(v + |v|) - \omega_{i-1,j}^{(n+1/2)}(v - |v|) - 2\omega_{i,j}^{(n+1/2)}(|v|)\}\right]$$

$$+ r\left[\omega_{i,j-1}^{(n+1)} + \omega_{i,j+1}^{(n+1)} - 2\omega_{i,j}^{(n+1)} + \frac{Reh}{2}\{\omega_{i,j-1}^{(n+1)}(u + |u|) - \omega_{i,j+1}^{(n+1)}(u - |u|) - 2\omega_{i,j}^{(n+1)}(|u|)\}\right]. \quad (21)$$

Alternating Direction Line Iterative Method (ADLI)

We propose in this paper a variation of the conventional ADI procedure. It consists of employing the line iterative method described above, but alternately in the x and y directions. Thus a complete iteration comprises one iteration in the row-direction, followed by a second one in the column direction. The equations for the method are the same as Eqs. (14–17). We call the method "the Alternating Direction Line Iterative (ADLI)" method.

Second-Order Procedure

When an iteration procedure converges linearly, it is sometimes possible to accelerate the convergence by using a second order procedure, such as Aitken's extrapolation technique [10]. In this method the value of the function is obtained by extrapolation from three successive iterates to minimize the error. The following is the equation for the method,

$$f = f^{(n+2)} - \frac{(f^{(n+2)} - f^{(n+1)})^2}{(f^{(n+2)} - 2f^{(n+1)} + f^{(n)})}. \quad (22)$$

The method is to be used along with some other scheme such as SOR and is effective when applied after a few initial iterates have commenced the process of convergence.

Results and Discussion

Employing each of the methods described in the previous section, the equations of motion are integrated for the cavity problem. The acceleration procedures are tested against the well established SOR method. The methods are also studied for their utility in solving the energy equation. The comparisons are based on the number of iterations, time required for a specified convergence, and "operational

counts" per iteration. The "operational counts" are calculated taking into account the number of multiplications and divisions per iteration. The computational times reported in this paper refer to times taken on an IBM 360/44 system with an on-line input–output facility. Single precision arithmetic is found to be adequate for all the computations in this investigation.

A second comparison is presented to establish the accuracy of the results obtained in this investigation. Theoretical error estimates are difficult to obtain for the class of coupled, quasilinear equations considered here. Hence, we compare our results with the ones available in the literature.

The results of these studies are discussed in this section.

Navier–Stokes Equations

We first consider the effect of the convergence criterion, ϵ, on the accuracy of the solution. It may be recalled that the convergence criterion defined in this work is a relative maximum one. It is the maximum, in the field, of the ratio of change in the values of the function between two successive iterates to the value of the function (see Eq. (11)). Thus this criterion is satisfied by stream and vorticity functions at all points in the field. In general, when the vorticity function satisfies the convergence criterion, the stream function converges to an ϵ far smaller than the one specified.

TABLE I

Effect of Convergence Criterion on the Accuracy of the Solution u-Velocity for $Re = 1000$, $h = 0.05$

	$X = 0.2$		$X = 0.5$		$X = 0.8$	
	Convergence Criterion		Convergence Criterion		Convergence Criterion	
	0.005	0.001	0.005	0.001	0.005	0.001
1.0	−1.0	−1.0	−1.0	−1.0	−1.0	−1.0
0.95	−0.49215	−0.49216	−0.52230	−0.51225	−0.32612	−0.32613
0.75	−0.07977	−0.07978	−0.16328	−0.16334	−0.03847	−0.03852
0.65	−0.04425	−0.04427	−0.09336	−0.09340	−0.02858	−0.02861
0.55	−0.01019	−0.01022	−0.01764	−0.01766	+0.00467	+0.00468
0.45	+0.03037	+0.03035	+0.05727	+0.05729	+0.05007	+0.05010
0.35	+0.09824	+0.09824	+0.14269	+0.14271	+0.09623	+0.09626
0.25	+0.20051	+0.20052	+0.25323	+0.25327	+0.12417	+0.12421
0.15	+0.26724	+0.26729	+0.33845	+0.33842	+0.10336	+0.10338
0.05	+0.16653	+0.16550	+0.21385	+0.21389	+0.03061	+0.03061
0	0	0	0	0	0	0

The choice of ϵ is governed by the accuracy desired for the physically observable variables and the computational effort. Table I shows the values of the velocity function at three sections in the cavity for two values of ϵ, namely, 0.005 and 0.001. These results have been obtained by SOR method for a Reynolds number of 1000 and a mesh size of 0.05. An examination of the table reveals that the maximum difference between the two sets of values occurs at $x = 0.2$ and $y = 0.55$, and is less than 0.3 %. The rest of the field shows differences only in the fourth or fifth significant figure. Thus, for this class of problems, an $\epsilon = 0.005$ produces results which are of sufficient accuracy for all practical purposes. This choice of ϵ results in significant savings in computational effort for complex geometries such as in [7]–[9], though the savings are not of much consequence for the present geometry.

We now turn to the comparison of various methods for solving the algebraic equations corresponding to the Navier–Stokes equations. These results are summarized in Table II. They have been computed for a Reynolds number of 1000 and a mesh size of 0.05. The Thomas algorithm [10] for a tridiagonal matrix is used for solving the subsystems in all the implicit methods. The methods have been arranged in the order of their convergence rates.

TABLE II
Comparison of Iterative Methods–Navier–Stokes Equations
$Re = 1000, h = 0.05$

Method	ADI $r_\psi = 1.6$ $r_\omega = 0.1$	Gauss–Seidel	ADLI (G–S)	LINE SOR $r_\psi = 1.6$ $r_\omega = 1.1$ Column	LINE SOR $r_\psi = 1.6$ $r_\omega = 1.1$ Row	SOR $r_\psi = 1.6$ $r_\omega = 1.1$	ADLI $r_\psi = 1.6$ $r_\omega = 1.1$	Aitken's extrapolation
Number of Iterations	80	268	132	123	85	83	28	—
Time (sec.)	107	96	80	77	54	37	34	—
Operational counts	9020	4200	7420	7820	7820	4600	7820	—

The table does not include any results for the second order procedure—the Aitken's extrapolation procedure. The procedure failed to converge for all combinations of (under or over) relaxation parameters. The failure is attributed to the coupled nature of the governing equations. Extrapolation of one function—say, the stream function—to minimize the error, results in an increased error in the other function. Use of the procedure on one function only (in particular the stream function) does not assure the convergence of the system either.

Before discussing the results in Table II, we will digress a little and discuss some interesting features observed during the present work while using the alternating direction implicit (ADI) method. The method is yet to be investigated throughly even for linear systems. It is known to result in loss of accuracy when applied to certain systems [10]. For the present system of equations, the ADI method with optimum relaxation parameters ($r_\psi = 1.6$, $r_\omega = 1.1$) converges only for small Reynolds numbers (say 100). As the Reynolds number is increased to 1000 the method fails to converge. This is attributed to the loss of diagonal dominance in the coefficient matrix as shown below. In the tridiagonal matrix for the column iteration, the coefficients have the form

$$-[1 + (|v| - v)\,Reh/2] \quad [2 + |v|\,Reh + 1/r] \quad -[1 + (|v| + v)\,Reh/2].$$

Since the off-diagonal terms involve addition of velocity components, the diagonal term loses dominance whenever the Reynolds number becomes large. However, this loss can be compensated by making the relaxation factor, r, small enough. A similar situation prevails for row iteration as well. The following discussion will reveal some of the undesirable consequences of the above factors.

The solution obtained by the ADI method with optimum relaxation parameters agrees well with that of the SOR method for a Reynolds number of 100, as can be observed from Table III. The table shows center line u-velocities obtained by the

TABLE III

Comparison of ADI and SOR Methods Center Line u-Velocity for $Re = 100$, $h = 0.05$

Method	SOR	ADI
Y	$r_\psi = 1.6$ $r_\omega = 1.1$	$r_\psi = 1.6$ $r_\omega = 1.1$
1.0	−1.0	−1.0
0.95	−0.66219	−0.66218
0.85	−0.22962	−0.22960
0.75	−0.02967	−0.09640
0.65	+0.09442	+0.09444
0.55	+0.17075	+0.17074
0.45	+0.19677	+0.19674
0.35	+0.18140	+0.18137
0.25	+0.14345	+0.14344
0.15	+0.09633	+0.09633
0.05	+0.03805	+0.03805
0	0.0	0.0

TABLE IV

Effect of the Value of Relaxation Parameter on The Accuracy of the ADI Method. Center Line u-Velocity for $Re = 1000$, $h = 0.05$

Method	SOR	ADI	
Y	$r_\psi = 1.6$ $r_\omega = 1.1$	$r_\psi = 1.6$ $r_\omega = 0.5$	$r_\psi = 1.6$ $r_\omega = 0.1$
1.0	−1.0	−1.0	−1.0
0.95	−0.51223	−0.51234	−0.51203
0.85	−0.21888	−0.21904	−0.21848
0.75	−0.16328	−0.16334	−0.16287
0.65	−0.09336	−0.09334	−0.09306
0.55	−0.01764	−0.01756	−0.01753
0.45	+0.05727	+0.05736	+0.05718
0.35	+0.14269	+0.14278	+0.14248
0.25	+0.25323	+0.25330	+0.25294
0.15	+0.33845	+0.33836	+0.33790
0.05	+0.21385	+0.21381	+0.21347
0	0	0	0
Number of iterations	83	115	80

two methods. An agreement upto 4 significant figures is obtained. Table IV shows centre-line u-velocities for a Reynolds number of 1000 and two values of r_ω, 0.5 and 0.1, with the ADI procedure. The solution obtained by SOR with optimum relaxation parameters is also included for comparison. It is seen that the ADI method, in general, agrees with the SOR result to only three significant figures. For $r_\omega = 0.1$, the convergence is faster than that for $r_\omega = 0.5$. However, the agreement with the SOR result is poorer for $r_\omega = 0.1$, when compared with the result obtained with $r_\omega = 0.5$. Thus the solution obtained by the ADI method seems to depend on the relaxation parameter (which is to be reduced in order to obtain convergence at higher Reynolds numbers). This makes the method unreliable.

Reverting to Table II, we will now consider the difference in convergence rates between the SOR and the line SOR methods. The line iterative methods are known to accelerate the convergence of linear equations [10]. However, for the problem under consideration, both row and column iterations do not give convergence rates better than SOR. Since the operational counts per iteration are much larger than the SOR, these methods involve large computational times. An interesting by-product of the study is the difference between the convergence rates of the column and row iterations. This arises because the maximum variation of stream and vorticity functions occurs along the column (y) direction. During a single

column iteration, the implicit solution adjusts the values only along one column while the others are assumed known. This is ineffective due to the neglect of large variations near the moving wall, in the adjacent columns. In the case of row iteration, since the iteration is carried out in the direction of maximum variation of the functions, it converges faster than column iteration.

The new method proposed in this paper, the alternating direction line iterative (ADLI) method, is tested for two sets of relaxation parameters, namely, one corresponding to the Gauss–Seidel scheme ($r_\psi = r_\omega = 1.0$) and the other corresponding to optimum relaxation parameters ($r_\psi = 1.6$, $r_\omega = 1.1$). The former reduces the number of iterations required for convergence to nearly half compared to that of the Gauss–Seidel method. The reduction in time is only 16% due to increase in operational counts. The latter procedure takes the fewest iterations for convergence of all the methods investigated in the present work. The number of iterations (= 28) reported in Table II implies 28 row iterations and 28 column iterations. The operational counts for the ADLI method is nearly double that of SOR. However, the time required for convergence is less than that of SOR. Thus the alternating direction line iterative method with optimum relaxation parameters turns out to be the most economical method of solving the finite difference equations of motion, both in the number of iterations and computer time.

In view of the above result, it was decided to investigate the method in greater detail. The program was run for different Reynolds numbers and mesh sizes and its performance was compared with that of SOR for the same parameter values. Table V shows the convergence rates and the time taken for the two methods with

TABLE V

Comparison of ADLI and SOR Methods–Computational Effort, Different Mesh Sizes, $Re = 1000$

Mesh size		1/10	1/20	1/30	1/40	1/50
Number of	SOR	51	83	219	375	669
Time (sec)	SOR	17	37	225	606	1591
	ADLI	17	34	183	667	1222
Percentage saving in computer Time with ADLI		0	8.1	18.7	−10.1	23.2

five different mesh sizes (0.1, 0.05, 0.033, 0.025 and 0.02), for a Reynolds number of 1000. The same relaxation parameters ($r_\omega = 1.1$, $r_\psi = 1.6$) were used for all mesh sizes. As the mesh size is reduced, the savings in computer time increase with the use of the ADLI procedure, the maximum saving occurring at $h = 0.02$. For

the mesh size $h = 0.025$, the ADLI method exhibits a poorer convergence rate than the SOR procedure and this is difficult to explain on the basis of the present work.

The solutions for different Reynolds numbers (10, 50, 100, 500, 1000, 2000 and 5000) are obtained with a mesh size of 0.05. The number of iterations, the time required for convergence and percentage savings in time when compared with that of SOR are tabulated in Table VI. The ADLI method results in substantial savings

TABLE VI

Comparison of ADLI and SOR Methods—Computational Effort, Different Reynolds Numbers, $h = 0.05$

Reynolds number		10	50	100	500	1000	2000	5000
Number of iterations	SOR	109	105	142	132	83	565	—
	ADLI	33	32	42	38	28	35	58
Time (sec)	SOR	49	47	63	59	37	265	—
	ADLI	40	38	51	46	34	43	75
Percentage saving in computer Time with ADLI		18.3	19.1	19.0	22.0	8.1	83.8	—

in computational time for the range of Reynolds numbers investigated. The saving is dramatic for higher Reynolds numbers: for $Re = 2000$ it takes less than one-sixth the time taken by SOR procedure; for $Re = 5000$, while the SOR fails to converge, the ADLI not only converges, but does so in a fairly short time. The behavior of SOR at high Reynolds numbers is attributed to severe oscillations that are encountered during computation. These results are somewhat different from those of Greenspan [2] who obtained solutions upto a Reynolds number of 10^5. The Greenspan work uses a modified form of SOR and involves a large number of parameters with attendant programming complexities. In addition, the boundary conditions are handled in a different way as explained earlier. The computational times are also very large. The ADLI method developed in the present work has not been tested beyond a Reynolds number of 5000 since laminar flows under such conditions are not physically realizable. In support of this, the experimental results of Pan and Acrivos [11] can be cited. According to their findings, the limit $Re \to \infty$ is attained at a value of $Re = 2700$ for a square cavity and further, flow instabilities set in at about $Re = 4000$ for cavities of large depths. Thus in the range of Reynolds numbers of practical consequence, the most efficient method of integrating the equations of motion for recirculating flows, is the "upwind" difference scheme coupled with a direct iteration technique and the ADLI proce-

dure. The ADLI method is successful because it combines the virtues of all the methods considered in this investigation with the exception of the second order procedure.

TABLE VII

Comparison of ADLI and SOR Methods—Accuracy
u-Velocity for $Re = 1000$, $h = 0.05$

	$X = 0.2$		$X = 0.5$		$X = 0.8$	
Y	SOR	ADLI	SOR	ADLI	SOR	ADLI
1.0	−1.0	−1.0	−1.0	−1.0	−1.0	−1.0
0.95	−0.49215	−0.49212	−0.51223	−0.51228	−0.32612	−0.32615
0.85	−0.13550	−0.13548	−0.21888	−0.21896	−0.02594	−0.02602
0.75	−0.07977	−0.07975	−0.16328	−0.16328	−0.03847	−0.03854
0.65	−0.04425	−0.04425	−0.09336	−0.09330	−0.02858	−0.02859
0.55	−0.01019	−0.01021	−0.01764	−0.01757	+0.00467	+0.00473
0.45	+0.03037	+0.03036	+0.05727	+0.05732	+0.05007	+0.05016
0.35	+0.09824	+0.09824	+0.14269	+0.14271	+0.09623	+0.09629
0.25	+0.20051	+0.19951	+0.25323	+0.25324	+0.12417	+0.12421
0.15	+0.26724	+0.26723	+0.33845	+0.33833	+0.10336	+0.10338
0.05	+0.16653	+0.16647	+0.21385	+0.21380	+0.03061	+0.03061
0	0	0	0	0	0	0

We next examine the ADLI method for its accuracy. Table VII shows the u-velocity components at three sections ($x = 0.2$, 0.5 and 0.8) of the cavity obtained by SOR and ADLI methods for $Re = 1000$ and $h = 0.05$. The two sets of results agree with each other to four significant figures.

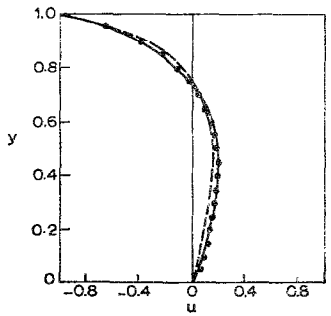

FIG. 5. Comparison of center line u-velocity. Reynolds number = 100. (——) $h = 1/50$ (Burggraf); (---) $h = 1/14$ (Mills); • $h = 1/20$ (Present).

Finally the solution obtained in this paper is compared with those available in the literature. No numerical comparisons are attempted since such data are not easily accessible to the authors. Graphical comparisons of centre line velocities for a Reynolds number of 100 obtained in the present investigation with those obtained by Mills [12] and Burggraf [13] are presented in Fig. 5. Both these investigators employ the central difference scheme. Mills uses the Liebmann iteration technique while Burggraf uses a modified relaxation procedure. The present solution obtained with a mesh size of 1/20 agrees very well with that of Burggraf, obtained with $h = 1/50$. The differences in the solutions between Mills and the present one are due to the coarse mesh ($h = 1/14$) used by Mills.

Energy Equation

The most successful procedure for solving the finite difference equations of motion is shown to be the ADLI method. This, along with the SOR procedure is considered, in this section, for the solution of the energy equation in finite difference form. The latter is included for purposes of comparison. The Aitken's extrapolation technique, which is known to yield very fast convergence for linear equations, is also studied. The relevant equations are similar to those of vorticity except for the additional parameter, Prandtl number. The results of these investigations are shown in Table VIII. The Prandtl number is unity, the Reynolds number is taken

TABLE VIII

Comparison of Iterative Methods—Energy Equation
$Re = 1000$, $Pr = 1.0$, $h = 0.05$.

Method	SOR $r_T = 1.1$	ADLI $r_T = 1.1$	Aitken's extrapolation $r_T = 1.1$
Number of iterations	122	54	38
Time (sec)	26	18	13

as 1000, the mesh size used is 1/20, and $\epsilon = 0.001$. The convergence criterion is chosen as 0.001 to maintain more or less the same accuracy as that of the stream function.

From the table, it is seen that the Aitken's extrapolation technique is the most efficient method for the solution of the energy equation, though it fails with the

Naiver–Stokes equations. This success must be attributed to the fact that we are solving a single linear equation in this case. The number of iterations required for convergence is less than one third of that of SOR, while the time is half of that of SOR.

The alternating direction line iterative method with optimum relaxation parameters is also more efficient than the SOR procedure. The number of iterations for ADLI is less than half of that of SOR, though the reduction in time is not of the same order due to increase in the operational counts.

Another comparison of interest is the convergence rate of these methods with various convergence criteria. Figure 6 shows the number of iterations required for

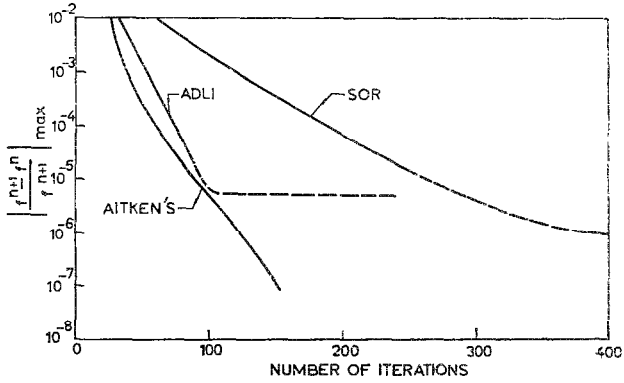

FIG. 6. Comparison of convergence rate of different methods–energy equation.

different convergence criteria. The SOR method starts oscillating (shown as dotted line in the figure) around 1×10^{-5} and the error never reduces below 0.5×10^{-5}. For the ADLI method the oscillations start at 4.2×10^{-5} and the minimum error observed is 7×10^{-6}. But in the case of Aitken's extrapolation, since we are extrapolating every time to minimize the error, it converges continuously to the minimum of 10^{-7} (which is about the minimum we can attain without roundoff noise in single precision arithmetic on the computer used in this work). The comparison of convergence rates is not presented for Navier–Stokes equations because of large changes in the maximum error in the functional values in the initial iterates. At this stage of iteration the error is also oscillatory in nature. After the system starts attaining convergent behavior, the maximum error in the vorticity function never reduces below 0.5×10^{-3} for ADLI and 0.1×10^{-3} for the SOR method. This is an obvious consequence of the coupled nature of the equations being solved.

1. R. COURANT, E. ISSACSON AND M. REES, On the solution of nonlinear hyperbolic differential equations by finite differences. *Comm. Pure Appl. Math.* **5** (1952), 243–255.
2. D. GREENSPAN, "Lectures on the Numerical Solution of Linear Singular and Nonlinear Differential Equations," Prentice Hall, Englewood Cliffs, NJ, 1968.
3. A. D. GOSMAN, W. M. PUN, A. K. RUNCHAL, D. B. SPALDING AND M. WOLFSHTEIN, "Heat and Mass Transfer in Recirculating Flows," Academic Press, New York, 1969.
4. E. R. G. ECKERT AND R. M. DRAKE, JR., "Analysis of Heat and Mass Transfer," McGraw-Hill, New York, 1972.
5. DAVID M. YOUNG, "Iterative Solution of Large Linear Systems," Academic Press, New York, 1971.
6. M. FRIEDMAN, Laminar flow in a channel with a step, *J. Eng. Math.* **6** (1972), 285–290.
7. M. NALLASAMY AND K. KRISHNA PRASAD, Numerical investigation of Laminar Heat transfer in step induced internal separated flows. Paper No. 7, Proc. of First National Heat and Mass Transfer Conference, Madras, India, 1971.
8. M. NALLASAMY AND K. KRISHNA PRASAD, Heat transfer to transpired laminar internal separated regions. Paper No. A-15, Proc. of Second National Heat and Mass Transfer Conference, Kanpur, India, 1973.
9. M. NALLASAMY AND K. KRISHNA PRASAD, Flow and heat transfer in internal laminar separated regions. To be published.
10. W. F. AMES, "Numerical Methods for Partial Differential Equations," Nelson, Edinburgh, 1969.
11. F. PAN AND A. ACRIVOS, Steady flow in rectangular cavities, *J. Fluid Mech.* **28** (1967), 643–655.
12. R. D. MILLS, Numerical solution of the viscous flow equations for a class of closed flows. *J. Roy. Aero. Soc.* **69** (1965), 714–718.
13. O. R. BURGGRAF, Analytical and numerical studies of the structure of steady separated flows. *J. Fluid Mech.* **24** (1966), 113–151.